Mesh dependence in PDE-constrained optimisation problems with an application in tidal turbine array layouts

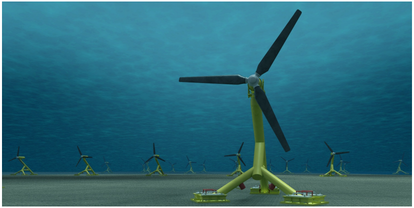
Tobias Schwedes

Imperial College London, University of Reading

7th October 2015

Energy production using tidal forces

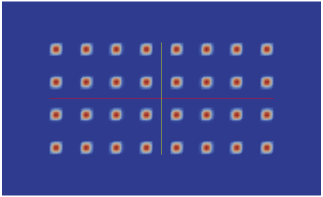
- Tidal turbines: Extract energy from tidal currents
- Industrially relevant scale: Arrays comprising dozens to hundreds of turbines
- Suitable sites: high peak flow rates as $P \propto u^3$



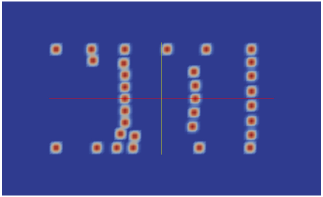
https://islayenergytrust.files.wordpress.com/2009/02/hs-array.jpg

Layout optimization for tidal turbine arrays

- Turbine placement affects the flow
- Optimizing locations of turbines has enormous impact on extracted power (Funke et al., 2014)
- OpenTidalFarm performs layout optimization by applying efficient gradient-based optimization algorithms.



$\Rightarrow$



+75% power

PDE-constraint optimization

Common structure:

$$\min_{d \in D} J(z(d), d), \qquad \text{(objective)}$$

subject to

$$h(d) \leq 0, \qquad \text{(inequality constraint)}$$

$$g(d) = 0, \qquad \text{(equality constraint)}$$

where

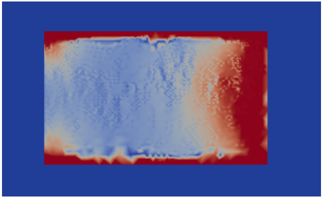
- $J : Z \times D \to \mathbb{R}$ is the objective functional
- $D \ni d$ is the control space
- $z : D \to Z$ is the operator that solves the PDE

$$F(z(d), d) = 0.$$

- D and Z are Hilbert spaces

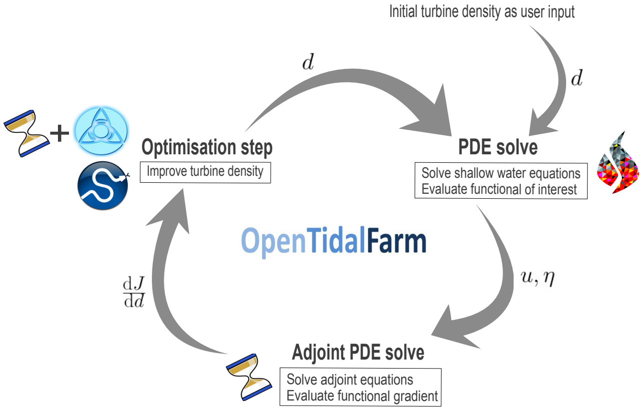
Continuous approach

- Turbine farm configuration represented by spatially varying density function, i.e. $D \ni d$ is a function space



- Advantages over discrete approach:
  - By integrating over optimised density, one obtains an approximation for the optimal number of turbines
  - Turbines not individually resolved $\Rightarrow$ lower mesh resolution still produces reasonable results

Optimisation loop

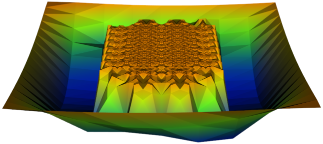


- $z = (u, \eta)$ solution of the shallow water equations

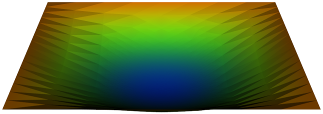
Gradient depends on inner product

- Computing $\mathrm{d}J/\mathrm{d}d$ is crucial for optimization
- Riesz-representation theorem: For a Hilbert space H, every linear functional (an element of H^*) is isomorphic to an element of H.
- The gradient is a Riesz-representation of $\mathrm{d}J/\mathrm{d}d$:

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}d}(d)\delta d &= \nabla J_1(d) \cdot \delta d \\ &= \big(\nabla J_2(d), \delta d\big)_{L^2} \\ &= \big(\nabla J_3(d), \delta d\big)_{H^1} \end{aligned}$$



Gradient in ℓ^2 inner product



Gradient in L^2 inner product

Which representation is to choose? Is it important?

- Naturally, $(\cdot, \cdot)_D$ corresponds to control space D
- Most implementations of optimisation methods assume $D = \mathbb{R}^n$
- What if $D \neq \mathbb{R}^n$? Particularly, what happens in the continuous approach?

Which representation is to choose? Is it important?

- Naturally, $(\cdot, \cdot)_D$ corresponds to control space D
- Most implementations of optimisation methods assume $D = \mathbb{R}^n$
- What if $D \neq \mathbb{R}^n$? Particularly, what happens in the continuous approach?

Intuitively: Disrespecting inner products is somehow inaccurate (think geometrically: angles, distances)

Question: What exactly are the drawbacks?

1-D continuous optimisation problem

(1) $$\min_{u \in L^2([0,1])} \left\{ f(u) = (1-u, 1-u)_{L^2} \right\}$$

$$\frac{\mathrm{d}f}{\mathrm{d}u}(u)(\cdot) = -(1-u, \cdot)_{L^2} \implies \nabla f_{L^2}(u) = -(1-u)$$

Continuous L^2 representation: Using steepest descent with exact line search with $u_0 = 0$, the minimum is found after one iteration!

Applying finite element discretisation $\implies$ (1) becomes

(2) $$\min_{\vec{u} \in \mathbb{R}^n} \left\{ f(\vec{u}) = \frac{1}{2}(\vec{1} - \vec{u})^T M (\vec{1} - \vec{u}) \right\}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\vec{u}}(\vec{u})(\cdot) = -\left((\vec{1} - \vec{u})^T M, \cdot\right)_{\ell^2} \implies \nabla f_{\ell^2}(\vec{u}) = -(\vec{1} - \vec{u})^T M$$

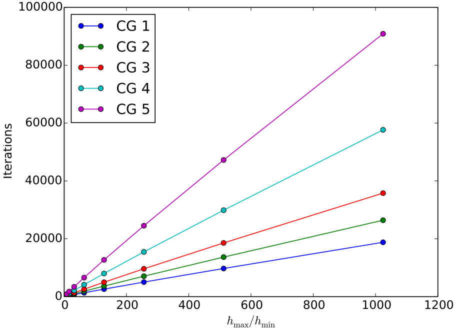
Gradient now contains scaling by mass matrix!

How many iterations k using ℓ^2 representation?

Analytically: Given a convergence threshold ε,

$$k \geq -\frac{3}{2}\log(2\varepsilon)\frac{h_{\max}}{h_{\min}} - \frac{1}{4}\log(2\varepsilon) \quad \left(\text{linear in } \frac{h_{\max}}{h_{\min}}\right)$$

Numerically:

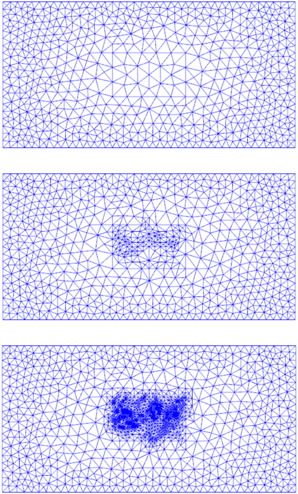


$\Longrightarrow$ Disrespecting inner product yields mesh-dependent convergence!

$\Longrightarrow$ Several hundred thousand iterations vs 1 !

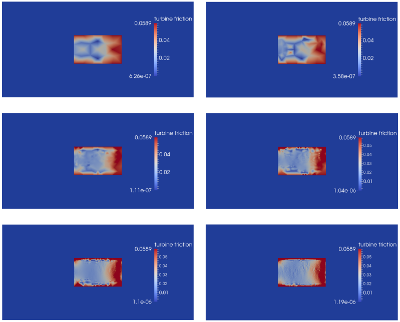
How does this relate to the continuous turbine optimisation problem?

Randomly refined meshes

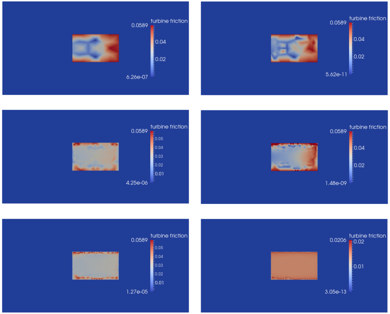


Two inner product representations for $\mathrm{d}J/\mathrm{d}d$

(a) L^2 representation

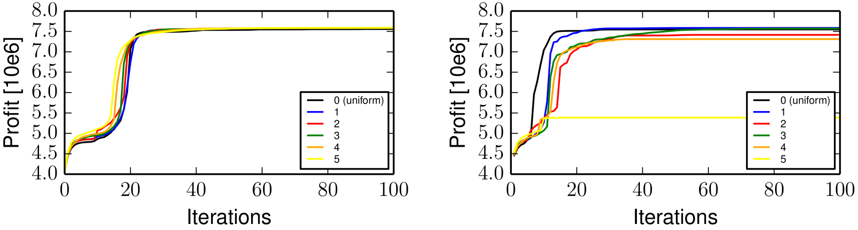


(b) ℓ^2 representation



Two inner product representations for $\mathrm{d}J/\mathrm{d}d$

(a) L^2 representation (b) ℓ^2 representation



$\Longrightarrow$ Choice of inner product may decide over economic viability!

$\Longrightarrow$ "Respect the inner product of the control space of your problem!"

Many thanks for your attention!

References:

- Funke, S. W., P.E. Farrell, M.D. Piggott, *"Tidal turbine array optimisation using the adjoint approach".* Renewable Energy 63 (2014)
- Funke, S. W., S. C. Kramer, M. D. Piggott, *"Design Optimisation and Resource Assessment for Tidal-Stream Renewable Energy Farms Using a New Continuous Turbine Approach".* preprint arXiv:1507.05795 (2015)
- OpenTidalFarm: Open-source software for simulating and optimising tidal turbine farms, http://opentidalfarm.org/
- Optizelle: Open-source software library designed to solve general purpose non-linear optimization problems, https://www.optimojoe.com/products/optizelle/

- Wathen, A. J., *Realistic Eigenvalue Bounds for the Galerkin Mass Matrix.* Journal of Numerical Analysis 7, 4 (1987)
- Graves, L. M., *Riemann Integration and Taylor's Theorem in general Analysis.* Transactions of the American Mathematical Society 29, 1 (1927)
- Byrd, R. H., P. Lu, J. Nocedal and C. Zhu. *A Limited Memory Algorithm for Bound Constrained Optimization.* SIAM J. Sci. Comput. 16 (1994)
- Tits, A. L., A. Wächter, S. Bakhtiari, J. T. Urban and C. T. Lawrence, *A Primal-dual Interior-point Method for Nonlinear Programming with Strong Global and Local Convergence Properties.* SIAM Journal on Optimization 14, 1 (2003)
- Hinze, M., R. Pinnau and M. Ulbrich. ,*Optimization with PDE Constraints.* Springer Science + Business Media B.V., 2009
- Pope, S. B., *Turbulent flows.* Cambridge University Press, 2000
- Nocedal, J. and S. Wright. *Numerical Optimization.* Springer Science+Business Media, LLC, 2006.
- Brenner, S. C. and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* 3rd Edition, Springer Science+Business Media, LLC, 2008